

Elastic fields of stationary and moving dislocations in three dimensional finite samples

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Abstract

Integral expressions are determined for the elastic displacement and stress fields due to stationary or moving dislocation loops in three dimensional, not necessarily isotropic, finite samples. A line integral representation is found for the stress field, thus satisfying the expectation that stresses should depend on the location of the dislocation loop, but not on the location of surfaces bounded by such loops that are devoid of physical significance.

In the stationary case the line integral representation involves a “vector potential” that depends on the specific geometry of the sample, through its Green’s function: a specific combination of derivatives of the elastic stress produced by the Green’s function appropriate for the sample is divergenceless, so it is the curl of this “vector potential”. This “vector potential” is explicitly determined for an isotropic half space and for a thin plate. Earlier specific results in these geometries are recovered as special cases.

In the non stationary case a line integral representation can be obtained for the time derivative of the stress field. This, combined with the static result, assures a line integral representation for the time dependent stress field.

1 Introduction

A general formula for the displacement and stress fields generated by a dislocation loop undergoing arbitrary motion in an infinite medium was obtained some years ago by Mura (1963). In this formulation, the stress field is written as a convolution of the medium’s impulse response with a source localized along the dislocation loop (i.e. independent of the loop’s slip plane). These formulae do not apply to the case of finite samples since they are derived assuming homogeneity in space, i.e. a dependence of the Green’s function in $\vec{x} - \vec{x}'$, a fact that is no longer true in a finite sample. The purpose of this paper is to present a generalization of these formulae to finite samples, as well as explicit forms for the geometries of a half space and thin plate in an isotropic elastic medium.

Motivation for this work comes partly from an ongoing project that attempts to understand recently observed dynamic instabilities for cracks in thin plates (Sharon, Gross and Fineberg, 1996, Boudet, Ciliberto and Steinberg, 1996) in terms of the interaction of the plate’s oscillations with the crack tip regarded as a continuous distribution of infinitesimally small dislocations (Lund, 1996). An intuitive thought behind this work was that the stress fields produced by stationary or moving dislocation loops in a finite sample should not depend on the slip planes chosen, and that indeed the

stresses will be continuous through the slip planes, as it is the case in an infinite medium. This is rigorously shown here: the expressions found for the stress fields involve only line integrals over the dislocation loops. In addition, Gosling and Willis (1994) have emphasized the importance of dealing with free surface effects in order to understand the strain relaxation within thin strained layers over a substrate such as are employed in the manufacture of high technology devices. They developed a line integral representation for the stresses due to an arbitrary dislocation in an isotropic half-space, in which the integrand can be constructed by an explicitly given algorithm. The resulting integral can be explicitly evaluated for a dislocation half-line. Line integral representations are computationally more efficient than the more easily derived surface integral representations.

In order to find the explicit form of the line integrals appearing in the stress fields when the loops are stationary, one should calculate a “vector potential”, that is, a vector quantity whose curl gives a combination of derivatives of the inhomogeneous portion of the Green’s function, appropriate to the finite sample in question. We have explicitly done this for two examples: an isotropic half space and a thin plate. In addition to the calculation of the “vector potential” in these two geometries we have explicitly worked out the case of a stationary screw dislocation perpendicular to the free surfaces. These two examples can also be obtained independently with the method of Eshelby and Stroh (1951), thus providing a check on the algebra. In a non stationary situation the displacement velocity field can be written in terms of line integrals, and from this one deduces that the time derivative of the stress field can be written in terms of them too. One finally deduces that the time dependent stress field can be written in terms of line integrals since one knows that the initial condition (stationary case) and its temporal derivative can both be written in terms of line integrals.

2 Displacement fields due to dislocations in finite samples

Here we derive a formula for the elastic displacement field produced in a finite sample by a moving dislocation loop, that is written as an integration over successive slip planes.

We consider small displacements $U_m(x, t)$ in an homogeneous elastic medium of density ρ and elastic constants C_{ijkl} . A general formula for the displacement, velocity and stress fields generated by a dislocation loop undergoing arbitrary motion in an infinite medium was obtained some years ago by Mura (1963). The displacement field is written as an integration over successive slip planes, where the displacement field has a discontinuity given by the Burgers vector b_i . The velocity and stress fields are written as a convolution of the medium’s impulse response with a source localized along the dislocation loop, i.e. these fields are independent of the choice of slip planes. In this section Mura’s formula for the displacement field is generalized to a finite sample.

The particle displacement obeys the wave equation:

$$\rho \frac{\partial^2 U_i}{\partial t'^2}(x', t') - C_{ijkl} \frac{\partial^2 U_k}{\partial x'_l \partial x'_j}(x', t') = 0, \quad (1)$$

and normal stress-free boundary conditions at the surfaces of the sample:

$$C_{ijkl} \frac{\partial U_k}{\partial x'_l}(x'_S, t) n_j(x'_S) = 0, \quad (2)$$

with x'_S a point on the surface and $n_j(x'_S)$ its outwardly-pointing normal vector. The medium’s impulse response is given by a Green’s function $G_{im}(x, x'; t - t')$, which is the displacement in

direction (i) evaluated at (x, t) produced by a localized impulse force in direction (m) applied at position x' and time t' . It is the solution of:

$$\rho \frac{\partial^2 G_{im}}{\partial t^2}(x, x'; t - t') - C_{ijkl} \frac{\partial^2 G_{km}}{\partial x_l \partial x_j}(x, x'; t - t') = \delta_{im} \delta(x - x') \delta(t - t'). \quad (3)$$

We have explicitly exhibited the x and x' dependence because the order in which they appear is important. If one defines as $\sigma_{ij}^{Gm}(x, x'; t - t')$ the elastic stress associated with the Green's function:

$$\sigma_{ij}^{Gm}(x, x'; t - t') \equiv C_{ijkl} \frac{\partial}{\partial x_k} G_{lm}(x, x'; t - t'), \quad (4)$$

the free surface boundary condition reads:

$$\sigma_{ij}^{Gm}(x_S, x'; t - t') n_j(x_S) = 0. \quad (5)$$

Note that, since the medium is bounded, there is homogeneity in time but not in space.

In order to find the displacement field due to a moving dislocation we start, following Mura (1963), with the following identity that holds due to a symmetry of the elastic constants tensor ($C_{ijkl} = C_{klij}$):

$$C_{ijkl} \frac{\partial U_k}{\partial x'_l}(x', t') \frac{\partial G_{mi}}{\partial x'_j}(x, x'; t - t') = C_{ijkl} \frac{\partial U_i}{\partial x'_j}(x', t') \frac{\partial G_{mk}}{\partial x'_l}(x, x'; t - t'). \quad (6)$$

A single dislocation loop is considered to be present. At the position of the loop itself the derivatives of the displacement field have singularities, and across the slip plane the displacement field has a discontinuity equal to the Burgers vector b_i . The identity in Eq. (6) is integrated over the volume of the elastic sample, excluding a thin tube around the dislocation loop, as well as a thin layer encompassing the slip plane. Using the reciprocity relation of the impulse response (Poruchikov, 1993):

$$G_{im}(x, x'; t - t') = G_{mi}(x', x; t - t'), \quad (7)$$

as well as Eqns. (1)-(5), and integrating by parts, the following identity results:

$$\begin{aligned} \int dS'_j C_{ijkl} \frac{\partial U_k}{\partial x'_l}(x', t') G_{mi}(x, x'; t - t') - \int dx' C_{ijkl} \frac{\partial^2 U_k}{\partial x'_l \partial x'_j}(x', t') G_{mi}(x, x'; t - t') = \\ \int dS'_j C_{ijkl} U_i(x', t') \frac{\partial G_{mk}}{\partial x'_l}(x, x'; t - t') - \int dx' C_{ijkl} U_i(x', t') \frac{\partial^2 G_{mk}}{\partial x'_l \partial x'_j}(x, x'; t - t'), \end{aligned} \quad (8)$$

where the surface integrals are carried out over the tube and layer just introduced, as well as over the sample boundary. The first term on the left vanishes on the surfaces of the sample due to the free surface boundary condition, Eq. (2), and it also vanishes on the slip plane due to continuity of the stress there. The first term on the right also vanishes on the surfaces of the sample due to the free surface boundary condition, Eq. (5), due to reciprocity, but it will give a contribution on the slip plane due to the discontinuity of U_i given by the Burgers vector b_i . The first terms on the left and on the right of Eq. (8) integrated over the small tube (of radius ϵ) around the dislocation loop vanish because of the short distance behaviour of particle displacement U_i (Lund, 1988).

Using the differential equations satisfied by U_i and G_{mi} one gets from Eq. (8):

$$\begin{aligned} -\rho \int dx' \frac{\partial^2 U_i(x', t')}{\partial t'^2} G_{mi}(x, x'; t - t') = b_i \int_{S(t')} dS'_j C_{ijkl} \frac{\partial G_{mk}}{\partial x'_l}(x, x'; t - t') \\ -\rho \int dx' \frac{\partial^2 G_{mi}}{\partial t'^2}(x, x'; t - t') U_i(x', t') + \delta(t - t') U_m(x, t'), \end{aligned} \quad (9)$$

with $S(t')$ the slip plane at time t' . Integrating over time t' , one obtains the expression for the displacement as an integral over the successive slip planes:

$$U_m(x, t) = -b_i \int_{-\infty}^{\infty} dt' \int_{S(t')} dS'_j C_{ijkl} \frac{\partial G_{mk}}{\partial x'_l}(x, x'; t - t') . \quad (10)$$

Using reciprocity, this can be written as

$$U_m(x, t) = -b_i \int_{-\infty}^{\infty} dt' \int_{S(t')} dS'_j \sigma_{ij}^{Gm}(x', x; t - t') . \quad (11)$$

Attention should be paid to the order in which the variables appear in this formula. In an analogous way, for the stationary case, the following displacement field is derived:

$$U_m(x) = -b_i \int_{S'} dS'_j \sigma_{ij}^{Gm}(x', x) , \quad (12)$$

with the static Green's function satisfying:

$$C_{ijkl} \frac{\partial^2}{\partial x_k \partial x_j} G_{lm}(x, x') = \frac{\partial}{\partial x_j} \sigma_{ij}^{Gm}(x, x') = -\delta_{im} \delta(x - x') , \quad (13)$$

and the free surface boundary condition:

$$\sigma_{ij}^{Gm}(x_S, x') n_j(x_S) = 0 . \quad (14)$$

Eq. (10) leads to null normal stresses at the free surfaces. Indeed, from Eq. (10) one gets

$$\begin{aligned} \sigma_{pq}(x, t) &= -b_i C_{pqnm} \int_{-\infty}^{\infty} dt' \int_{S(t')} dS'_j C_{ijkl} \frac{\partial^2 G_{mk}}{\partial x_n \partial x'_l}(x, x'; t - t') \\ &= -b_i C_{ijkl} \int_{-\infty}^{\infty} dt' \int_{S(t')} dS'_j \frac{\partial \sigma_{pq}^{Gk}}{\partial x'_l}(x, x'; t - t') . \end{aligned} \quad (15)$$

Since $\sigma_{pq}^{Gk}(x_S, x'; t - t') n_q(x_S) = 0$, this implies that $\sigma_{pq}(x_S, t) n_q(x_S) = 0$, as it should.

3 Stress fields due to dislocation loops in finite samples

3.1 Stationary case

Starting from Eq. (12), the stress field produced by the static dislocation loop is:

$$\sigma_{pq}(x) = -b_i C_{pqlm} \int_{S'} dS'_j \frac{\partial}{\partial x_l} \sigma_{ij}^{Gm}(x', x) . \quad (16)$$

In the infinite medium case, space homogeneity easily leads to a line integral representation (Mura, 1963). In order to obtain a line integral representation for this stress field in the finite case, we add and subtract to the integrand the term

$$\frac{\partial}{\partial x'_l} \sigma_{ij}^{Gm}(x', x) ,$$

which is easy to turn into a line integral, and we define a tensor:

$$f_{ij}^{lm}(x', x) \equiv \frac{\partial}{\partial x_l} \sigma_{ij}^{Gm}(x', x) + \frac{\partial}{\partial x'_l} \sigma_{ij}^{Gm}(x', x), \quad (17)$$

which is divergenceless. Indeed, from Eq. (13),

$$\frac{\partial}{\partial x'_j} f_{ij}^{lm}(x', x) = -\delta_{im} \left(\frac{\partial}{\partial x_l} + \frac{\partial}{\partial x'_l} \right) \delta(x - x') = 0. \quad (18)$$

This means that $f_{ij}^{lm}(x', x)$ can be written as the curl of a “vector potential” $A_s^{ilm}(x', x)$:

$$f_{ij}^{lm}(x', x) = \epsilon_{jrs} \frac{\partial}{\partial x'_r} A_s^{ilm}(x', x), \quad (19)$$

and it is possible to write the stress field, Eq. (16), as:

$$\sigma_{pq}(x) = -b_i C_{pqilm} \int_{S'} dS'_j \left\{ f_{ij}^{lm}(x', x) - \frac{\partial \sigma_{ij}^{Gm}}{\partial x'_l}(x', x) \right\}. \quad (20)$$

When x is not in the slip plane S' the following is true:

$$\int dS'_j \frac{\partial \sigma_{ij}^{Gm}}{\partial x'_l}(x', x) = \epsilon_{vjl} \epsilon_{vsr} \int dS'_s \frac{\partial}{\partial x'_r} \sigma_{ij}^{Gm}(x', x), \quad (21)$$

so that we finally have that the stress field can be expressed as a line integral along the dislocation loop:

$$\sigma_{pq}(x) = b_i C_{pqilm} \int_{L'} dl'_s \left\{ \epsilon_{sjl} \sigma_{ij}^{Gm}(x', x) - A_s^{ilm}(x', x) \right\}, \quad (22)$$

as promised, with $A_s^{ilm}(x', x)$ a “vector potential” to be determined for each specific sample by use of Eqns. (17) and (19). This means that σ_{ij}^{Gm} , the Green’s function for the finite sample, must be known. Its determination, however, is of course independent of the particular dislocation loop that might be under consideration.

Eqn. (22) is similar to Eqn. (12) of Gosling and Willis (1994). The difference between those two expressions resides in the fact that Gosling and Willis (1994) split the integrand into the Green’s function for an infinite medium plus “image” forces that annul the tractions at the free surface.

3.2 Non stationary case

Since the problem is homogeneous under time translations it is possible to obtain an expression for particle velocity, that involves an integral along the dislocation loop only. Indeed, from Eq. (11):

$$\frac{\partial U_m}{\partial t}(x, t) = b_i \int_{-\infty}^{\infty} dt' \left\{ \frac{\partial}{\partial t'} \left[\int_{S(t')} dS'_j \sigma_{ij}^{Gm}(x', x; t - t') \right] - \int_{\frac{dS}{dt'}(t')} dS'_j \sigma_{ij}^{Gm}(x', x; t - t') \right\}. \quad (23)$$

The first term is zero since the Green’s function vanishes at $t - t' = \pm\infty$. Since:

$$\int_{dS(t')/dt'} dS'_j = \epsilon_{jpq} \int_{L(t')} dl'_q V_p(x', t'),$$

where $L(t')$ is the dislocation loop bounding the slip plane $S(t')$, $V_p(x', t')$ is the loop's local velocity and $\epsilon_{j pq}$ the completely antisymmetric tensor in three dimensions; we then have:

$$\frac{\partial U_m}{\partial t}(x, t) = -b_i \int_{-\infty}^{\infty} dt' \int_{L(t')} dl'_q \sigma_{ij}^{Gm}(x', x; t - t') \epsilon_{j pq} V_p(x', t'). \quad (24)$$

From this line integral expression for the displacement velocity field, one obtains an expression for the partial time derivative of the stress field that involves line integrals over the dislocation loop:

$$\frac{\partial}{\partial t} \sigma_{ln}(x, t) = -C_{lnkm} b_i \int_{-\infty}^{\infty} dt' \int_{L(t')} dl'_q \frac{\partial}{\partial x_k} \sigma_{ij}^{Gm}(x', x; t - t') \epsilon_{j pq} V_p(x', t'). \quad (25)$$

This result, together with the result for a static loop (considered as an initial condition), mean that the time dependent stress field can be written in terms of integration of line integrals over the dislocation loop, and it is independent of the choice of slip planes, as it should. Explicitly, we are thinking on the generally valid formula:

$$\sigma_{ln}(x, t) = \sigma_{ln}(x, t = 0) + \int_0^t dt' \frac{\partial \sigma_{ln}}{\partial t}(x, t'), \quad (26)$$

with both terms on the right hand side having a line integral representation.

A general result that derives from Eq. (25) is that segments of the dislocation loop on the free surfaces do not contribute to $\partial \sigma_{ln} / \partial t$: in those segments $dl' \times \vec{V}$ is parallel to the normal \hat{n} to the free surface, and then the boundary condition satisfied by σ_{ij}^{Gm} at the free surfaces cancels that contribution.

4 Examples of explicit “Vector potentials” for the stationary case

In two geometries, a half space and a thin plate, and considering an isotropic elastic medium we have determined explicitly the form of the “vector potentials” appearing in the general line integral representation of Eq. (22) for the stress field generated by a dislocation in a finite sample. The derivation will be explained in the following, with further details to be found in Appendices I and II.

The elastic medium fills the half space $z \geq 0$, and the thin plate the space between $-h \leq z \leq h$, where z is the coordinate whose direction is perpendicular to the free surfaces of these geometries (we will call \vec{R} a coordinate in the planes parallel to the free surfaces). In both geometries the appropriate Green's function depends on $\vec{R} - \vec{R}'$ because of homogeneity in those directions, meaning that only differentiation with respect to z and z' can give a non zero result for the “vector potentials” of Eq. (19); in other words, only the component $l = z$ of A_s^{ilm} is non zero. The equation to be solved for A_s^{izm} is then:

$$\frac{\partial}{\partial z} \sigma_{ij}^{Gm}(z', z; \vec{R}' - \vec{R}) + \frac{\partial}{\partial z'} \sigma_{ij}^{Gm}(z', z; \vec{R}' - \vec{R}) = \epsilon_{jrs} \frac{\partial}{\partial x'_r} A_s^{izm}(z', z; \vec{R}' - \vec{R}). \quad (27)$$

We have obtained the Green's function appropriate for these geometries, satisfying Eqs. (13) and (14), using the method of Chishko (1989), who found the time dependent Green's function for a thin plate. The Green's functions are written as the infinite medium Green's function ($G_{im}^{(0)}$)

minus a term (H_{im}) that satisfies the homogeneous equilibrium equation and that cancels the normal stresses produced by the infinite medium's Green's function at the free surfaces:

$$G_{im}(z, z'; \vec{R} - \vec{R}') = G_{im}^{(0)}(x - x') - H_{im}(z, z'; \vec{R} - \vec{R}') . \quad (28)$$

This strategy differs somewhat from the one employed by Gosling and Willis (1994), who used a similar split not for the Green's function, but for a combination of gradients of the Green's function.

The representation used for H_{im} for the half space is the following:

$$H_{im}(z, z'; \vec{R} - \vec{R}') = - \int d\vec{R}'' G_{in}^{(h)}(z; \vec{R} - \vec{R}'') \sigma_{nz}^{(0)m}(-z'; \vec{R}'' - \vec{R}') , \quad (29)$$

and a similar form for the thin plate:

$$H_{im}(z, z'; \vec{R} - \vec{R}') = \sum_{a=1,2} \int dS_a'' G_{in}^{(p)}(z', \xi^{(a)}; \vec{R}' - \vec{R}'') \sigma_{nz}^{(0)m}(\xi^{(a)} - z; \vec{R}'' - \vec{R}') , \quad (30)$$

where the index $(a) = 1, 2$ represents the free surfaces at $z'' = \xi^{(a)} = \pm h$, and $dS_a'' = \pm d\vec{R}''$. We have introduced the surface Green's functions $G_{in}^{(h),(p)}$ for the half space and the thin plate: they are the displacements in direction (i) produced by a localized impulse in direction (n) applied at the free surface. They satisfy, respectively, the homogeneous equilibrium equations:

$$C_{ijpq} \frac{\partial^2}{\partial x_p \partial x_j} G_{qn}^{(h)}(z; \vec{R} - \vec{R}') = 0 \quad (31)$$

$$C_{ijpq} \frac{\partial^2}{\partial x_p \partial x_j} G_{qn}^{(p)}(z, \xi; \vec{R} - \vec{R}') = 0 \quad (32)$$

($\xi = \pm h$), and they are subject to the boundary conditions:

$$\sigma_{iz}^{(h)n}(z = 0; \vec{R} - \vec{R}') = -\delta_{in} \delta(\vec{R} - \vec{R}') \quad (33)$$

$$\sigma_{iz}^{(p)n}(\xi^{(b)}, \xi^{(a)}; \vec{R} - \vec{R}') = \pm \delta^{(a)(b)} \delta_{in} \delta(\vec{R} - \vec{R}') . \quad (34)$$

It is possible to solve for the Fourier components (in the plane parallel to the free surface) of the Green's functions $G_{ln}^{(h)}(z; \vec{R} - \vec{R}')$ ($g_{ln}^{(h)}(z|\vec{k})$), and $G_{ln}^{(p)}(z, \xi; \vec{R} - \vec{R}')$ ($g_{ln}^{(p)}(z, \xi|\vec{k})$):

$$\begin{aligned} g_{ln}^{(h)}(z|\vec{k}) &= \int d\vec{R} e^{i\vec{k} \cdot \vec{R}} G_{ln}^{(h)}(z; \vec{R}) \\ g_{ln}^{(p)}(z|\vec{k}) &= \int d\vec{R} e^{i\vec{k} \cdot \vec{R}} G_{ln}^{(p)}(z; \vec{R}) , \end{aligned} \quad (35)$$

and they are given in Appendices I and II. From Eqs. (29) and (30), one obtains in Fourier space:

$$\sigma_{ij}^{H(h)m}(z', z|\vec{k}) = -\sigma_{ij}^{(h)n}(z'|\vec{k}) \sigma_{nz}^{(0)m}(-z|\vec{k}) \quad (36)$$

$$\sigma_{ij}^{H(p)m}(z', z|\vec{k}) = \sum_{\xi} (\pm) \sigma_{ij}^{(p)n}(z', \xi|\vec{k}) \sigma_{nz}^{(0)m}(\xi - z|\vec{k}) , \quad (37)$$

with $\xi = \pm h$. Expressions for $\sigma_{ij}^{H(h)m}(z'|\vec{k})$ and $\sigma_{nz}^{(0)m}(z - z'|\vec{k})$ are shown also in Appendix I, and for $\sigma_{ij}^{H(p)m}(z', \xi|\vec{k})$ in Appendix II.

Since $\sigma_{ij}^{(0)m}$ depends on $(z' - z)$, only the term σ_{ij}^{Hm} contributes to the determination of the “vector potentials” in Eq. (27). If $a_s^{im}(z', z; \vec{R}' - \vec{R})$ is the “vector potential” associated with σ_{ij}^{Hm} , namely

$$\sigma_{ij}^{Hm}(z', z; \vec{R}' - \vec{R}) = \epsilon_{jrs} \frac{\partial a_s^{im}}{\partial x'_r}(z', z; \vec{R}' - \vec{R}), \quad (38)$$

then the “vector potential” A_s^{izm} that satisfies Eq. (27) is given by:

$$A_s^{izm}(z', z; \vec{R}' - \vec{R}) = -(\frac{\partial}{\partial z} + \frac{\partial}{\partial z'}) a_s^{im}(z', z; \vec{R}' - \vec{R}). \quad (39)$$

For the sake of brevity, we will present the stress function tensor associated with σ_{ij}^{Hm} , from which a_s^{im} and further A_s^{izm} can be determined by differentiation. The stress function tensor χ_{qn}^m is defined through a “double curl”:

$$\sigma_{ij}^{Hm}(z', z; \vec{R}' - \vec{R}) = -\epsilon_{ipq} \epsilon_{jln} \frac{\partial^2 \chi_{qn}^m}{\partial x'_p \partial x'_l}(z', z; \vec{R}' - \vec{R}), \quad (40)$$

a form that assures that σ_{ij}^{Hm} , symmetric in (ij) , is divergenceless in the indices i and j , as it should. It is easy to see that:

$$a_s^{im}(z', z; \vec{R}' - \vec{R}) = -\epsilon_{ipq} \frac{\partial \chi_{qs}^m}{\partial x'_p}(z', z; \vec{R}' - \vec{R}). \quad (41)$$

In Fourier space the stress function tensor for the isotropic half space geometry is:

$$\begin{aligned} \chi_{\eta\nu}^z(z', z|\vec{k}) &= -\frac{e^{-k(z+z')}}{2\gamma^2(\gamma^2-1)k^4} \{(\gamma^2-2)[(\gamma^2+1)+2(\gamma^2-1)kz]k_\eta k_\nu \\ &\quad -(\gamma^2-1)[\gamma^2+(\gamma^2-1)kz+(\gamma^2+1)kz'+2(\gamma^2-1)k^2zz']q_\eta q_\nu\} \\ \chi_{\eta\nu}^\beta(z', z|\vec{k}) &= \frac{ie^{-k(z+z')}}{2\gamma^2(\gamma^2-1)k^5} \{\gamma^2(\gamma^2-1)q_\beta(q_\eta k_\nu + q_\nu k_\eta) \\ &\quad +(\gamma^2-1)[1-(\gamma^2-1)kz+(\gamma^2+1)kz'-2(\gamma^2-1)k^2zz']k_\beta q_\eta q_\nu \\ &\quad -(\gamma^2-2)[(\gamma^2+1)-2(\gamma^2-1)kz]k_\beta k_\eta k_\nu\}, \end{aligned} \quad (42)$$

where $q_\nu \equiv \epsilon_{\nu\delta} k_\delta$, greek indices mean coordinates x or y , and $\gamma^2 \equiv c_l^2/c_t^2$ is the ratio between the longitudinal and transverse sound velocities in an isotropic medium. These formulae were obtained using the explicit forms for the Fourier components $\sigma_{ij}^{Hm}(z', z|\vec{k})$ that are written in Appendix I, Eq. (61), and solving for a stress function tensor that has only components $\chi_{\eta\nu}^m \neq 0$ in order to reproduce σ_{ij}^{Hm} . Similarly for a plate:

$$\begin{aligned} \chi_{\eta\nu}^z(z', z|\vec{k}) &= \sum_\xi (\pm) \frac{e^{-k|\xi-z|}}{4\gamma^2(\gamma^2-1)k^4} \{(\gamma^2-1)[(1+(\gamma^2-1)k|\xi-z|)(kz'\phi_4 - k\xi\phi_3) \\ &\quad -s(\xi-z)(\gamma^2+(\gamma^2-1)k|\xi-z|)(\phi_3 + k\xi\phi_1 - kz'\phi_2)]q_\eta q_\nu \\ &\quad +(\gamma^2-2)[s(\xi-z)(\gamma^2+(\gamma^2-1)k|\xi-z|)\phi_3 + (1+(\gamma^2-1)k|\xi-z|)\phi_1]k_\eta k_\nu\} \\ \chi_{\eta\nu}^\beta(z', z|\vec{k}) &= \sum_\xi (\pm) \frac{ie^{-k|\xi-z|}}{4\gamma^2(\gamma^2-1)k^5} \{(\gamma^2-1)[(1-(\gamma^2-1)k|\xi-z|)(\phi_3 + k\xi\phi_1 - kz'\phi_2) \end{aligned}$$

$$\begin{aligned}
& -s(\xi - z)(\gamma^2 - (\gamma^2 - 1)k|\xi - z|)(kz'\phi_4 - k\xi\phi_3)]k_\beta q_\eta q_\nu \\
& -(\gamma^2 - 2)[s(\xi - z)(\gamma^2 - (\gamma^2 - 1)k|\xi - z|)\phi_1 + (1 - (\gamma^2 - 1)k|\xi - z|)\phi_3]k_\beta k_\eta k_\nu \\
& +s(\xi - z)\gamma^2(\gamma^2 - 1)\left(\frac{\cosh(kz')}{\sinh(k\xi)} + \frac{\sinh(kz')}{\cosh(k\xi)}\right)q_\beta(q_\eta k_\nu + q_\nu k_\eta)\}.
\end{aligned} \tag{43}$$

5 Example: a screw dislocation perpendicular to the free surfaces of a half space and a thin plate

Eshelby and Stroh (1951) obtained the elastic fields corresponding to a stationary screw dislocation perpendicular to the free surfaces of a thin plate. From their method it is easy to obtain the corresponding fields for a half space. As examples of the use of the general line integral representation for the stresses presented in this paper, Eq. (22), we rederive these exact results. In the following we go through the main steps in the calculation.

5.1 A screw dislocation in a half space

Using the method in Eshelby and Stroh (1951), a simple calculation gives the displacement and stress fields, in polar coordinates, due to a screw dislocation lying along the z axis perpendicular to the free surface of a half space $z > 0$:

$$\begin{aligned}
U_z(\theta) &= \frac{b}{2\pi}\theta \\
U_\theta(R, z) &= \frac{b}{2\pi} \int_0^\infty \frac{dk}{k} e^{-kz} J_1(kR) \\
\sigma_{z\theta}(R, z) &= \frac{\mu b}{2\pi R} - \frac{\mu b}{2\pi} \int_0^\infty dk e^{-kz} J_1(kR) \\
\sigma_{r\theta}(R, z) &= -\frac{\mu b}{2\pi} \int_0^\infty dk e^{-kz} J_2(kR),
\end{aligned} \tag{44}$$

with $J_1(z)$ and $J_2(z)$ the Bessel functions of order one and two respectively, and μ the shear modulus.

Now we describe how these results can be obtained as a special case of the general formula, Eq. (22). When applied to this case it reads:

$$\sigma_{pq}(x) = \sigma_{pq}^{(1)}(x) + \sigma_{pq}^{(2)}(x), \tag{45}$$

where:

$$\sigma_{pq}^{(1)}(z, \vec{R}) \equiv bC_{pq\gamma m} \epsilon_{\eta\gamma} \int_0^\infty dz' \sigma_{z\eta}^{Gm}(z', z; -\vec{R}) \tag{46}$$

$$\sigma_{pq}^{(2)}(z, \vec{R}) \equiv -bC_{pqmz} \int_{-C} dR'_\beta A_\beta^{zzm}(z' = 0, z; \vec{R}' - \vec{R}), \tag{47}$$

with $-C$ a curve that closes the dislocation loop on the free surface, and $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$.

First we focus on the first term $\sigma_{pq}^{(1)}(x)$. For this term the segment on the free surface doesn't contribute due to the boundary condition: $\sigma_{zi}^{Gm}(x'_S, x) = 0$. In Fourier space Eq. (46) becomes:

$$\sigma_{pq}^{(1)}(z, -\vec{k}) = bC_{pq\gamma m} \epsilon_{\eta\gamma} \int_0^\infty dz' \sigma_{z\eta}^{Gm}(z', z|\vec{k}). \quad (48)$$

After doing the integration with the expressions in Appendix I, one obtains $\sigma_{zz}^{(1)}(z|\vec{k}) = 0$, and:

$$\sigma_{z\alpha}^{(1)}(z|\vec{k}) = -ib\mu \frac{\epsilon_{\alpha\eta} k_\eta}{k^2}. \quad (49)$$

Going back to real space, and in polar coordinates, we get

$$\sigma_{z\theta}^{(1)}(z|\vec{R}) = \frac{ib\mu}{(2\pi)^2} \int d\vec{k} \frac{e^{-i\vec{k}\cdot\vec{R}}}{k} (\sin\theta \sin\alpha + \cos\theta \cos\alpha), \quad (50)$$

where α is the polar angle associated with \vec{k} , and θ the angle of \vec{R} with respect to some fixed axis. Defining $\phi = \alpha - \theta$, one gets effectively an integral over ϕ , which is:

$$\sigma_{z\theta}^{(1)}(z|\vec{R}) = \frac{ib\mu}{(2\pi)^2} \int_0^\infty dk \int_{-\pi}^\pi d\phi e^{-ikR \cos\phi} \cos\phi = \frac{b\mu}{2\pi} \int_0^\infty dk J_1(kR) = \frac{\mu b}{2\pi R}, \quad (51)$$

i.e. it reproduces the first exact term of $\sigma_{z\theta}(R, z)$ in Eq. (44). Similarly it can be shown that $\sigma_{zr}^{(1)}(R, z) = 0$. Also, this type of term is the only contributing to the calculation of $\sigma_{r\theta}(R, z)$, and by a similar integration one gets the exact result in Eq. (44).

Now we focus on the second term $\sigma_{pq}^{(2)}(x)$. For this term, only the segment on the surface contributes. From expressions in Appendix I, we see that the “vector potential” $A_\beta^{zz\delta}$ is zero at the free surface, and that $A_\beta^{zz\delta}$ is given in Fourier space by:

$$A_\beta^{zz\delta}(z' = 0, z; \vec{k}) = \frac{e^{-kz}}{k^2} k_\beta \epsilon_{\delta\eta} k_\eta. \quad (52)$$

Now, the integral $\int_{-C} dR'_\beta A_\beta^{zz\delta}(x', x)$ appearing in Eq. (47) is independent of the choice of curve $-C$ on the free surface because the component z of the curl of $A_j^{zz\delta}$ with respect to the index j is zero on the free surface: indeed, $ik_\gamma \epsilon_{\gamma\beta} A_\beta^{zz\delta}(z' = 0, z|\vec{k}) = 0$. Thus, the expression for $\sigma_{z\delta}^{(2)}(x)$ in Eq. (47) becomes:

$$\sigma_{z\delta}^{(2)}(x) = -\frac{b\mu}{(2\pi)^2} \int_{-\pi}^\pi d\alpha \int_{-C} d\vec{R}' \cdot \vec{k} \int_0^\infty \frac{dk}{k} e^{-ik|\vec{R}' - \vec{R}| \cos\phi} \epsilon_{\delta\eta} k_\eta e^{-kz}, \quad (53)$$

with α the angle of \vec{k} and ϕ the angle between \vec{k} and $(\vec{R}' - \vec{R})$. The curve C was chosen in a convenient way. First, it starts at $\vec{R}' = 0$, which coincides with the position of the “vertical” segment of the dislocation. Then it continues in a semicircle around the point \vec{R} (there $|\vec{R}' - \vec{R}| = R$). After that, it continues on a straight line parallel to the direction of the vector \vec{R} ($\vec{R}' - \vec{R} = \lambda \vec{R}$, $\lambda \in (1, \infty)$). In this way, the result from both of these segments adds up to the result:

$$\sigma_{z\theta}^{(2)}(R, z) = -\frac{b\mu}{2\pi} \int_0^\infty dk e^{-kz} J_1(kR), \quad (54)$$

which is the second term of the exact result in Eq. (44), as it remained to be proven.

5.2 A screw dislocation in a plate

Eshelby and Stroh (1951) obtained the exact displacement and stress fields, in polar coordinates, due to a screw dislocation along the z axis perpendicular to the free surfaces of a plate (the plate exists for $-h < z < h$):

$$\begin{aligned} U_z(\theta) &= \frac{b}{2\pi}\theta \\ U_\theta(R, z) &= -\frac{b}{2\pi} \int_0^\infty \frac{dk}{k} \frac{\sinh(kz)}{\cosh(kh)} J_1(kR) \\ \sigma_{z\theta}(R, z) &= \frac{\mu b}{2\pi R} - \frac{\mu b}{2\pi} \int_0^\infty dk \frac{\cosh(kz)}{\cosh(kh)} J_1(kR) \\ \sigma_{r\theta}(R, z) &= \frac{\mu b}{2\pi} \int_0^\infty dk \frac{\sinh(kz)}{\cosh(kh)} J_2(kR). \end{aligned} \quad (55)$$

The verification that these results follow as a special case of Eq. (22) is analogous to the half-space case: one needs the “vector potential” $A_\beta^{zz\delta}$ evaluated at the free surfaces, which in Fourier space is, from Appendix I,

$$A_\beta^{zz\delta}(\pm h, z; \vec{k}) = -\frac{k_\beta \epsilon_{\delta\nu} k_\nu}{2k^2} \left\{ \pm \frac{\cosh(kz)}{\cosh(kh)} + \frac{\sinh(kz)}{\sinh(kh)} \right\}. \quad (56)$$

The terms $\sigma_{pq}^{(1)}(x)$ and $\sigma_{pq}^{(2)}(x)$ from Eq. (22) become:

$$\sigma_{pq}^{(1)}(z, \vec{R}) \equiv b C_{pq\gamma m} \epsilon_{\eta\gamma} \int_{-h}^h dz' \sigma_{z\eta}^{Gm}(z', z; -\vec{R}) \quad (57)$$

$$\sigma_{pq}^{(2)}(z, \vec{R}) \equiv 2b C_{pqmz} \frac{1}{(2\pi)^2} \int d\vec{k} \int_C dR'_\beta k_\beta e^{-i\vec{k} \cdot (\vec{R}' - \vec{R})} \frac{\epsilon_{\delta\nu} k_\nu \cosh(kz)}{2k^2 \cosh(kh)}. \quad (58)$$

The necessary integrations are done similarly as for the half space case (the curve C is the same), and the only non zero components of the stress that are obtained are those given by Eq. (55). ($\sigma_{pq}^{(1)}(x)$ contributes the first term of $\sigma_{z\theta}$, and $\sigma_{pq}^{(2)}(x)$ the second term of $\sigma_{z\theta}$).

6 Discussion

We have given a line integral representation for the stresses of an arbitrary dislocation loop in an arbitrary, not necessarily isotropic, three dimensional, finite elastic body. This representation is given in terms of a “vector potential”, for whose computation the Green’s function for the elastic body in question must be known. This, in general, is not the case, but use of approximate expressions will yield approximations to the corresponding stresses. At any rate, we have provided explicit forms of the said “vector potentials” for a half space and for a thin plate. Previous results for the stresses due to a screw dislocation in a thin plate (Eshelby and Stroh, 1951), as well as easily obtained stresses for a screw dislocation perpendicular to the surface of a half space, were recovered as special cases.

Gosling and Willis (1994) have provided an alternative line integral representation for the stress due to an arbitrary dislocation in an isotropic half-space. Our formulation is more general, admittedly at the price of introducing the Green’s function for a finite elastic sample, a quantity that is in general hard to compute.

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Appendix I: Green's function of a half space

We write the Fourier components of the surface Green's function appropriate for a half space, $g_{ij}^{(h)}(z|\vec{k})$ (they are the solutions in Fourier space of Eq. (31) subject to the boundary condition given by Eq. (33)) :

$$\begin{aligned}
g_{zz}^{(h)}(z|\vec{k}) &= \frac{e^{-kz}}{2\mu(\gamma^2 - 1)k} \{\gamma^2 + (\gamma^2 - 1)kz\} \\
g_{z\beta}^{(h)}(z|\vec{k}) &= \frac{ik_\beta e^{-kz}}{2\mu(\gamma^2 - 1)k^2} \{1 + (\gamma^2 - 1)kz\} \\
g_{\beta z}^{(h)}(z|\vec{k}) &= \frac{ik_\beta e^{-kz}}{2\mu(\gamma^2 - 1)k^2} \{-1 + (\gamma^2 - 1)kz\} \\
g_{\alpha\beta}^{(h)}(z|\vec{k}) &= \frac{e^{-kz}}{2\mu(\gamma^2 - 1)k^3} \{2(\gamma^2 - 1)q_\alpha q_\beta + [\gamma^2 - (\gamma^2 - 1)kz]k_\alpha k_\beta\}, \tag{59}
\end{aligned}$$

where $q_\alpha \equiv \epsilon_{\alpha\nu} k_\nu$ is a vector perpendicular to k_ν ($\epsilon_{12} = 1$, $\epsilon_{21} = -1$). The Fourier components of $\sigma_{iz}^{(0)m}(z - z'|\vec{k})$ are:

$$\begin{aligned}
\sigma_{zz}^{(0)z}(z - z'|\vec{k}) &= -\frac{s(z - z')}{2\gamma^2} e^{-k|z - z'|} \{\gamma^2 + (\gamma^2 - 1)k|z - z'|\} \\
\sigma_{\beta z}^{(0)z}(z - z'|\vec{k}) &= -\frac{ik_\beta}{2\gamma^2 k} e^{-k|z - z'|} \{1 + (\gamma^2 - 1)k|z - z'|\} \\
\sigma_{zz}^{(0)\delta}(z - z'|\vec{k}) &= \frac{ik_\delta}{2\gamma^2 k} e^{-k|z - z'|} \{1 - (\gamma^2 - 1)k|z - z'|\} \\
\sigma_{\beta z}^{(0)\delta}(z - z'|\vec{k}) &= -\frac{s(z - z')}{2\gamma^2 k^2} e^{-k|z - z'|} \{\gamma^2 q_\beta q_\delta + [\gamma^2 - (\gamma^2 - 1)k|z - z'|]k_\beta k_\delta\}. \tag{60}
\end{aligned}$$

Doing the calculation in Eq. (36), one gets the Fourier components $\sigma_{ij}^{Hm}(z', z|\vec{k})$:

$$\begin{aligned}
\sigma_{zz}^{Hz}(z', z|\vec{k}) &= \frac{e^{-k(z+z')}}{2\gamma^2} \{\gamma^2 + (\gamma^2 - 1)kz + (\gamma^2 + 1)kz' + 2(\gamma^2 - 1)k^2 zz'\} \\
\sigma_{z\beta}^{Hz}(z', z|\vec{k}) &= -\frac{ik_\beta}{2\gamma^2 k} e^{-k(z+z')} \{1 + (\gamma^2 - 1)kz - (\gamma^2 + 1)kz' - 2(\gamma^2 - 1)k^2 zz'\} \\
\sigma_{\alpha\beta}^{Hz}(z', z|\vec{k}) &= \frac{e^{-k(z+z')}}{2\gamma^2(\gamma^2 - 1)k^2} \{(\gamma^2 - 2)[(\gamma^2 + 1) + 2(\gamma^2 - 1)kz]q_\alpha q_\beta \\
&\quad + (\gamma^2 - 1)[(\gamma^2 + 2) + 3(\gamma^2 - 1)kz - (\gamma^2 + 1)kz' - 2(\gamma^2 - 1)k^2 zz']k_\alpha k_\beta\} \\
\sigma_{zz}^{H\delta}(z', z|\vec{k}) &= \frac{ik_\delta}{2\gamma^2 k} e^{-k(z+z')} \{1 - (\gamma^2 - 1)kz + (\gamma^2 + 1)kz' - 2(\gamma^2 - 1)k^2 zz'\}
\end{aligned}$$

$$\begin{aligned}
\sigma_{\beta z}^{H\delta}(z', z|\vec{k}) &= \frac{e^{-k(z+z')}}{2\gamma^2 k^2} \{ \gamma^2 q_\delta q_\beta + [\gamma^2 - (\gamma^2 - 1)kz - (\gamma^2 + 1)kz' + 2(\gamma^2 - 1)k^2 z z'] k_\delta k_\beta \} \\
\sigma_{\alpha\beta}^{H\delta}(z', z|\vec{k}) &= \frac{i e^{-k(z+z')}}{2\gamma^2 (\gamma^2 - 1) k^3} \{ (\gamma^2 - 2)[(\gamma^2 + 1) - 2(\gamma^2 - 1)kz] k_\delta q_\alpha q_\beta + \\
&\quad (\gamma^2 - 1)[2\gamma^2 + 1 - 3(\gamma^2 - 1)kz - (\gamma^2 + 1)kz' + 2(\gamma^2 - 1)k^2 z z'] k_\delta k_\alpha k_\beta \\
&\quad + \gamma^2 (\gamma^2 - 1) q_\delta [k_\alpha q_\beta + k_\beta q_\alpha] \} .
\end{aligned} \tag{61}$$

Finally, we derive the form of the Fourier components of the “vector potential” $A_\beta^{zzz}(z', z|\vec{k})$ and $A_\beta^{zz\delta}(z', z|\vec{k})$, necessary for the integration of Eq. (47). From Eqs. (41) and (42):

$$a_\beta^{zz}(z', z|\vec{k}) = -iq_\eta \chi_{\eta\beta}^z = -\frac{iq_\beta e^{-k(z+z')}}{2\gamma^2 k^2} [\gamma^2 + (\gamma^2 - 1)kz + (\gamma^2 + 1)kz' + 2(\gamma^2 - 1)k^2 z z'] , \tag{62}$$

and from Eq. (39):

$$A_\beta^{zzz}(z', z|\vec{k}) = -\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z'}\right) a_\beta^{zz}(z', z|\vec{k}) = -\frac{2iq_\beta}{\gamma^2 k} k z' e^{-k(z+z')} [1 + (\gamma^2 - 1)kz] , \tag{63}$$

or $A_\beta^{zzz}(z' = 0, z|\vec{k}) = 0$. Similarly:

$$\begin{aligned}
a_\beta^{z\delta}(z', z|\vec{k}) &= -iq_\eta \chi_{\eta\beta}^\delta = \frac{e^{-k(z+z')}}{2\gamma^2 k^3} \{ \gamma^2 q_\delta k_\beta \\
&\quad + [1 - (\gamma^2 - 1)kz + (\gamma^2 + 1)kz' - 2(\gamma^2 - 1)k^2 z z'] k_\delta q_\beta \} ,
\end{aligned} \tag{64}$$

and from Eq. (39):

$$A_\beta^{zz\delta}(z', z|\vec{k}) = -\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z'}\right) a_\beta^{z\delta}(z', z|\vec{k}) = \frac{e^{-k(z+z')}}{\gamma^2 k^2} \{ \gamma^2 k_\beta q_\delta + 2[\gamma^2 - (\gamma^2 - 1)kz] k z' k_\delta q_\beta \} , \tag{65}$$

or $A_\beta^{zz\delta}(z' = 0, z|\vec{k}) = \exp(-kz) k_\beta q_\delta / k^2$.

Appendix II: Green's function of a thin plate

We write the Fourier components of the surface Green's function appropriate for a thin plate, $g_{ij}^{(p)}(z|\vec{k})$ (they are the solution to Eqs. (32) and (34)):

$$\begin{aligned}
g_{zz}^{(p)}(z, \xi|\vec{k}) &= \frac{1}{4\mu(\gamma^2 - 1)k} \{ \gamma^2 \phi_2(z|\xi) - (\gamma^2 - 1)kz \phi_3(z|\xi) + (\gamma^2 - 1)k\xi \phi_4(z|\xi) \} \\
g_{z\beta}^{(p)}(z, \xi|\vec{k}) &= \frac{ik_\beta}{4\mu(\gamma^2 - 1)k^2} \{ (\gamma^2 - 1)kz \phi_1(z|\xi) - (\gamma^2 - 1)k\xi \phi_2(z|\xi) - \phi_4(z|\xi) \} \\
g_{\beta z}^{(p)}(z, \xi|\vec{k}) &= -\frac{ik_\beta}{4\mu(\gamma^2 - 1)k^2} \{ (\gamma^2 - 1)k\xi \phi_1(z|\xi) - (\gamma^2 - 1)kz \phi_2(z|\xi) - \phi_3(z|\xi) \} \\
g_{\alpha\beta}^{(p)}(z, \xi|\vec{k}) &= \frac{k_\alpha k_\beta}{4\mu(\gamma^2 - 1)k^3} \{ (\gamma^2 - 1)kz \phi_4(z|\xi) - (\gamma^2 - 1)k\xi \phi_3(z|\xi) + \gamma^2 \phi_1(z|\xi) \} \\
&\quad + (\pm) \frac{q_\alpha q_\beta}{2\mu k^3} \left[\frac{\cosh(kz)}{\sinh(k\xi)} + \frac{\sinh(kz)}{\cosh(k\xi)} \right] ,
\end{aligned} \tag{66}$$

where $\xi = \pm h$, and the functions $\phi_1(z|\xi)$ to $\phi_4(z|\xi)$ are defined in the following way:

$$\begin{aligned}
\phi_1(z|\xi) &= \frac{1}{\delta_s} \cosh(kz) \cosh(k\xi) + \frac{1}{\delta_a} \sinh(kz) \sinh(k\xi) \\
\phi_2(z|\xi) &= \frac{1}{\delta_s} \sinh(kz) \sinh(k\xi) + \frac{1}{\delta_a} \cosh(kz) \cosh(k\xi) \\
\phi_3(z|\xi) &= \frac{1}{\delta_s} \cosh(kz) \sinh(k\xi) + \frac{1}{\delta_a} \sinh(kz) \cosh(k\xi) \\
\phi_4(z|\xi) &= \frac{1}{\delta_s} \sinh(kz) \cosh(k\xi) + \frac{1}{\delta_a} \cosh(kz) \sinh(k\xi), \tag{67}
\end{aligned}$$

with:

$$\begin{aligned}
\delta_s &\equiv \sinh(kh) \cosh(kh) + kh \\
\delta_a &\equiv \sinh(kh) \cosh(kh) - kh. \tag{68}
\end{aligned}$$

Doing the calculation in Eq. (37), one gets the Fourier components $\sigma_{ij}^{Hm}(z', z|\vec{k})$:

$$\begin{aligned}
\sigma_{zz}^{Hz}(z', z|\vec{k}) &= \sum_{\xi} (\pm) \frac{e^{-k|\xi-z|}}{4\gamma^2} \{ (1 + (\gamma^2 - 1)k|\xi - z|)(kz'\phi_4(z'|\xi) - k\xi\phi_3) \\
&\quad - s(\xi - z)(\gamma^2 + (\gamma^2 - 1)k|\xi - z|)(\phi_3 + k\xi\phi_1 - kz'\phi_2) \} \\
\sigma_{z\beta}^{Hz}(z', z|\vec{k}) &= - \sum_{\xi} (\pm) \frac{ik_{\beta}e^{-k|\xi-z|}}{4\gamma^2k} \{ s(\xi - z)(\gamma^2 + (\gamma^2 - 1)k|\xi - z|)(kz'\phi_3(z'|\xi) - k\xi\phi_4) \\
&\quad + (1 + (\gamma^2 - 1)k|\xi - z|)(\phi_4 + kz'\phi_1 - k\xi\phi_2) \} \\
\sigma_{\alpha\beta}^{Hz}(z', z|\vec{k}) &= - \sum_{\xi} (\pm) \frac{e^{-k|\xi-z|}}{4\gamma^2(\gamma^2 - 1)} \{ q_{\alpha}q_{\beta}(\gamma^2 - 2) \times \\
&\quad [s(\xi - z)(\gamma^2 + (\gamma^2 - 1)k|\xi - z|)\phi_3 + (1 + (\gamma^2 - 1)k|\xi - z|)\phi_1] \\
&\quad + k_{\alpha}k_{\beta}(\gamma^2 - 1)[s(\xi - z)(\gamma^2 + (\gamma^2 - 1)k|\xi - z|)(\phi_3 + kz'\phi_2 - k\xi\phi_1) \\
&\quad + (1 + (\gamma^2 - 1)k|\xi - z|)(2\phi_1 + kz'\phi_4 - k\xi\phi_3)] \} \\
\sigma_{zz}^{H\delta}(z', z|\vec{k}) &= \sum_{\xi} (\pm) \frac{ik_{\delta}e^{-k|\xi-z|}}{4\gamma^2k} \{ (1 - (\gamma^2 - 1)k|\xi - z|)(\phi_3 + k\xi\phi_1 - kz'\phi_2) \\
&\quad - s(\xi - z)(kz'\phi_4 - k\xi\phi_3)(\gamma^2 - (\gamma^2 - 1)k|\xi - z|) \} \\
\sigma_{z\beta}^{H\delta}(z', z|\vec{k}) &= \sum_{\xi} (\pm) \frac{e^{-k|\xi-z|}}{4\gamma^2k^2} \{ k_{\delta}k_{\beta}[(1 - (\gamma^2 - 1)k|\xi - z|)(k\xi\phi_4 - kz'\phi_3) \\
&\quad - s(\xi - z)(\phi_4 + kz'\phi_1 - k\xi\phi_2)(\gamma^2 - (\gamma^2 - 1)k|\xi - z|)] \\
&\quad - (\pm)q_{\delta}q_{\beta}s(\xi - z)\gamma^2[\frac{\sinh(kz')}{\sinh(k\xi)} + \frac{\cosh(kz')}{\cosh(k\xi)}] \} \\
\sigma_{\alpha\beta}^{H\delta}(z', z|\vec{k}) &= \sum_{\xi} (\pm) \frac{e^{-k|\xi-z|}}{4\gamma^2(\gamma^2 - 1)k^3} \{ ik_{\delta}k_{\alpha}k_{\beta}(\gamma^2 - 1) \times \\
&\quad [(1 - (\gamma^2 - 1)k|\xi - z|)(\phi_3 + kz'\phi_2 - k\xi\phi_1) + \\
&\quad s(\xi - z)(\gamma^2 - (\gamma^2 - 1)k|\xi - z|)(2\phi_1 + kz'\phi_4 - k\xi\phi_3)] + ik_{\delta}q_{\alpha}q_{\beta}(\gamma^2 - 2) \times
\end{aligned}$$

$$\begin{aligned}
& [(1 - (\gamma^2 - 1)k|\xi - z|)\phi_3 + s(\xi - z)(\gamma^2 - (\gamma^2 - 1)k|\xi - z|)\phi_1] \\
& + (\pm)is(\xi - z)\gamma^2(\gamma^2 - 1)\left[\frac{\cosh(kz')}{\sinh(k\xi)} + \frac{\sinh(kz')}{\cosh(k\xi)}\right]q_\delta[k_\alpha q_\beta + k_\beta q_\alpha]\},
\end{aligned} \tag{69}$$

with $s(\xi - z) \equiv \text{sign}(\xi - z)$. Finally, we write the Fourier components of the “vector potential” evaluated at the free surfaces, necessary for the integration of Eq. (58):

$$\begin{aligned}
A_\beta^{zzz}(z' = \pm h, z|\vec{k}) &= 0 \\
A_\beta^{zz\delta}(z' = \pm h, z|\vec{k}) &= -\frac{k_\beta q_\delta}{2k^2} \left\{ \pm \frac{\cosh(kz)}{\cosh(kh)} + \frac{\sinh(kz)}{\sinh(kh)} \right\}.
\end{aligned} \tag{70}$$

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